

Minimal Length in Quantum Mechanics via Modified Heisenberg Algebra

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Abstract

In order to investigate some of the physical consequences of the existence of a minimal length, we propose to include it in the framework of Quantum Mechanics by modifying the Heisenberg algebra, so that a minimum non-zero value for the uncertainty Δx emerges, which, being a limitation to the localizability of particles, plays the desired role of a *minimal length*. The Hilbert space of the theory must be modified accordingly, and the changes are not merely quantitative. On the contrary, our main result is that the familiar concept of “position measurement” must be reformulated, as well as other concepts related to it, and we present a proposal for such reformulation. We also calculate the Casimir force resulting from the proposed algebra.

1 Introduction

It is a remarkable fact that, despite the radical differences (not only formal, but in content as well) between the various approaches to quantum gravity, all of them seem to coincide in one prediction: the existence of a *minimal length*, that is, a length scale below which the very concept of a length measurement loses its meaning.

In String Theory, it was shown [1, 2] that it is impossible to probe spacetime at distances below the characteristic length of the strings, because the behaviour of scattering between one-dimensional objects is fundamentally different from that between point-particles — and “any measurement in string theory involves string scattering” [3]. Indeed, while in QFT the resolution of a length measurement is inversely proportional to the momentum transferred in the process, thus allowing us to probe smaller spacetime regions by increasing the momentum of the incident particles, in String Theory there is a critical point in this momentum transfer above which the distance probed by the string starts increasing, rather than decreasing, so that “in no instance is the resolution smaller than the string length” [2]. It is as if the position-momentum uncertainty relation had the form

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$$\Delta x \Delta p \gtrsim \frac{\hbar}{2} + \alpha (\Delta p)^2, \quad (1)$$

where α is related to the characteristic length of the strings.

Loop Quantum Gravity, another popular attempt to quantize the gravitational interaction, tackles this problem from a completely different angle: instead of modifying the structure of matter, as String Theory does, it takes as a starting point the application of the canonical quantization formalism to Ashtekar variables of General Relativity [4]. The geometry of Spacetime becomes a dynamical entity, being described by a quantum state, and, since the area and volume of spacetime regions can be written in terms of Ashtekar's variables, they become quantum operators. It can then be shown [5, 6, 7, 8, 9] that their spectra is quantized, and that they have a minimum value — a result considered a great triumph of the theory. The rigorous calculations for the length operator are much more complicated, but seem to indicate that there is also a minimum length [10], as it is expected by the results of these other geometrical operators [11], which indicate that geometry itself is quantized.

To this “coincidence” of prediction between such radically different theories, we should also add that some qualitative arguments involving the principles of Quantum Mechanics and General Relativity also lead to the existence of a minimal observable length [12]. Let us consider, for example, an experiment in which we probe the content of a certain region of Spacetime through the scattering of some incident particles, which is what happens in modern large accelerators. According to the uncertainty principle of Quantum Mechanics, to probe smaller distances we need incident particles with larger momenta. But, in General Relativity, momentum is a source of gravitational field, and, therefore, in trying to probe smaller distances we increase the gravitational field created by the incident particles, which acts on the measured system and increasingly disturbs its trajectory, increasing (rather than decreasing) the uncertainty in its measurement — or, even worse, may create a black hole, resulting in no measurement at all.

All these facts suggest that a minimal observable length is an intrinsic effect of Quantum Gravity, and, for this reason, it is worth of some further investigation. In this work, we do not discuss the existence or the probable origins of a minimal length. Instead, we assume it as existent, and we try to introduce it *ad hoc* in the formalism of Quantum Mechanics in order to investigate the consequences of its existence, which concepts lose their meaning in its presence, and which new concepts eventually emerge. We do this by modifying the uncertainty relation in order to introduce a non-zero minimal value for Δx , which, being a limitation for the localizability of the physical systems, acts as a minimal length.

Our approach is similar to the one in [13], in which, aiming to recover the uncertainty relations (1), the authors propose a commutation relation of the form

$$[\hat{X}, \hat{P}] = i\hbar(1 + \beta \hat{P}^2), \quad (2)$$

and then conduct a thorough investigation on the structure of the associated Hilbert Space. It can be shown that $\Delta x \geq \hbar\sqrt{\beta}$, i.e., there is a minimum value for the uncertainty in position. We emphasize, however, that (1) is expressed only to first order of the minimum length parameter, and nothing prevents the existence of higher order terms.

Our proposal is to extend (2) adding terms of higher order of β , aiming to recover the dispersion relation

$$\hbar k(p) = \int_0^p e^{-\beta q^2} dq, \quad (3)$$

postulated in [14] to show that a minimal length regularizes the calculation of the Casimir Energy. To see that this dispersion relation is indeed related to this effect, note that in the limit $p \rightarrow \infty$ we have $k = 2\pi/\lambda \rightarrow \sqrt{\pi}/2\sqrt{\beta}$, so that λ is always greater than a minimum value. Also, note that when $\beta \rightarrow 0$ we have $\hbar k \rightarrow p$, which is the usual dispersion relation of Quantum Mechanics.

However, since the algebra of the operators is the very starting point of the formalism of Quantum Mechanics, the most fundamental way to introduce a minimal length in this formalism is by modifying this commutation relation of the quantum operators — and this is what led us to make this choice.

The article is organized in the following way. In section 2 we present our proposal for the Heisenberg algebra and show that it indeed leads us to a $\Delta x_{min} > 0$, which is our goal. Since we modify the algebraic structure on which the theory is based, we must rework the details resulting therefrom. This we do in section 3, where we investigate the structure of the Hilbert Space associated to the algebra we propose. One striking feature of this modified formalism is the absence of the position representation, since the eigenvectors of the position operator do not belong to the space of physical states anymore. This poses many conceptual problems related to how to recover information on the localization of the system in space, questions which we tackle in section 4. Another consequence of a minimal length worth being mentioned is that, because it extinguishes the notion of local interactions, it acts as a natural regulator parameter in QFT — and this is already another good reason to try to understand what quantum mechanics should look like in the presence of this feature. In section 5, we show how this regulator effect acts in the calculations of the Casimir Effect. We present our conclusions in section 6.

2 Minimal Length in Quantum Mechanics

One way to introduce a minimal length in Quantum Mechanics is to modify the uncertainty relations so that a non-zero minimum value for Δx emerges, which, being a limitation on the localizability of a system, plays this desired role. Since

$$\Delta x \Delta p \geq \frac{|\langle [\hat{X}, \hat{P}] \rangle|}{2}, \quad (4)$$

a modification of the uncertainty relations requires a modification in the algebra of the operators, and this implies in a complete reformulation of the structure of the theory.

In this work, we try to follow the approach of Kempf, Mangano, and Mann (KMM) [13], but with an algebra that leads to (3), to wit,

$$[\hat{X}, \hat{P}] = i\hbar e^{\beta \hat{P}^2}. \quad (5)$$

Note that, to first order of β (the parameter related to the minimal length) we recover (2).

The first thing we ought to do is to show that we indeed obtain a minimum value for Δx , which is our goal in the first place. From (4) and (5) we have

$$\Delta x \Delta p \geq \frac{\hbar}{2} \langle e^{\beta \hat{P}^2} \rangle. \quad (6)$$

Now, the so-called Jensen's Inequality [15] guarantees that, if ϕ is a real convex function, then $\langle \phi(\hat{X}) \rangle \geq \phi(\langle \hat{X} \rangle)$. Since e^{x^2} satisfies these requirements, it follows that

$$\Delta x \Delta p \geq \frac{\hbar}{2} e^{\beta \langle \hat{P}^2 \rangle} = \frac{\hbar}{2} e^{\beta \langle \hat{P} \rangle^2} e^{\beta (\Delta p)^2}. \quad (7)$$

The function

$$\frac{e^{\beta(\Delta p)^2}}{\Delta p}$$

has a minimum of $\sqrt{2\beta e}$, hence

$$\Delta x \geq \hbar\sqrt{\beta}\sqrt{\frac{e}{2}}e^{\beta(\hat{P})^2} \geq \hbar\sqrt{\beta}\sqrt{\frac{e}{2}} , \quad (8)$$

that is, (5) implies the existence of a $\Delta x_{min} \geq \hbar\sqrt{\beta}\sqrt{\frac{e}{2}}^1$.

3 Hilbert Space

The next step in this (re)development of the formalism is to build the Hilbert Space \mathcal{H} to which the physical states belong.

An important aspect of minimal length theories is that eigenstates of the position operator are no longer physical states, since for these states $\Delta x = 0 < \Delta x_{min}$. Consequently, we cannot work with the position representation $|x\rangle$ anymore. In “ordinary Quantum Mechanics”² we also had $|x\rangle \notin \mathcal{H}$, for these vectors have infinite norm³, but in this case we could work around the problem by defining $|x\rangle$ as a limit of the sequence of states $|\xi_x^{\Delta x}\rangle$ — for which $\langle\hat{X}\rangle = x$ and whose uncertainty is Δx — when $\Delta x \rightarrow 0$ ⁴. In our present case, not even that is possible, since we cannot take the limit $\Delta x \rightarrow 0$.

In the absence of the position representation, the simplest alternative is to work with the momentum representation $|p\rangle$ ⁵. The \hat{X} and \hat{P} operators in this representation are given by

$$\begin{aligned} \langle p|\hat{P}|\psi\rangle &= p\psi(p), \\ \langle p|\hat{X}|\psi\rangle &= i\hbar e^{\beta p^2} \frac{\partial}{\partial p}\psi(p) , \end{aligned} \quad (9)$$

where the exponential factor in the definition of \hat{X} is required so that (5) be satisfied. Note that this implies in a modification of the Schrödinger equation, which means the introduction of the parameter β affects the evolution of every quantum system.

These operators are hermitean under the inner product

$$\langle\phi|\psi\rangle = \int_{-\infty}^{\infty} e^{-\beta p^2} \phi^*(p)\psi(p)dp , \quad (10)$$

and from this we see that the measure of momentum space is modified, i.e.,

$$dp \rightarrow e^{-\beta p^2} dp .$$

¹In [16], this result was shown using a numerical method.

²We use this term to refer to Quantum Mechanics without a minimal length.

³The theory is already “warning us” about the problems related to the concept of perfect localizability.

⁴Rigorously speaking, this limit is also ill defined, since it suffers from the same problems the vectors $|x\rangle$ do. However, we can use this procedure of taking a limit to define $\langle x|\hat{A}|\psi\rangle \equiv \lim_{\Delta x \rightarrow 0} \langle \xi_x^{\Delta x}|\hat{A}|\psi\rangle$ for any operator \hat{A} and any vector $|\psi\rangle$. That is, we first evaluate $\langle \xi_x^{\Delta x}|\hat{A}|\psi\rangle$ and, then, take the limit, which is well-defined for an adequate choice of the sequence $|\xi_x^{\Delta x}\rangle$.

⁵When both representations are absent, due to a minimal uncertainty also in the momenta, an alternative is to work with the Bargmann-Fock representation. A detailed study of such case can be seen in [17].

This modification is not unexpected. The existence of a minimal length implies in an upper limit to the wave-number k , as we have already seen, so that any integral in k -space has finite integration limits. The same does not happen in momentum space, which is unlimited. Therefore, in a change of integration variables from k to p , the Jacobian must be a decreasing function that compensates this difference in the integration limits.

Finding another representation to the Hilbert Space is far from being the only difficulty to be overcome. Since the eigenvectors $|x\rangle$ of \hat{X} do not belong to \mathcal{H} anymore, this operator ceases to be an observable, even though it is still hermitean⁶. This immediately raises questions like: what is the meaning of the expression “position measurement” in this formalism (if it still has any meaning, since the operator associated to any measurable quantity must be an observable)? And how could we recover information on the propagation of the system in space? For example, how could we calculate the probability of finding the system in a certain spatial region?

These questions cannot simply be ignored and left unanswered, for three main reasons. First, because the formalism we now develop must coincide with the results of ordinary QM in the limit $\beta \rightarrow 0$, and, therefore, there must be some notion of spatial localization of the quantum system in this modified QM that, in this limit, gives the definition with which we are familiar. Secondly, the value of Δx associated to each state plays a central role in our work, since it is through a Δx_{min} that we introduce a minimal length in the theory; thus, it is necessary to guarantee that such Δx exists, that it makes sense even when \hat{X} ceases to be an observable, and to interpret its physical meaning in this case. Finally, because to understand some of the conceptual changes imposed by the existence of a minimal length is precisely one of the goals of this work.

3.1 Maximally Localized States

To answer these questions, we will use the concept of “maximal localization states” introduced in [13], which are nothing else than states for which $\Delta x = \Delta x_{min}$, i.e., states $|\psi_{\xi}^{ml}\rangle$ obeying

$$\begin{aligned}\langle\psi_{\xi}^{ml}|(\Delta\hat{X})^2|\psi_{\xi}^{ml}\rangle &= (\Delta x_{min})^2 \\ \langle\psi_{\xi}^{ml}|\hat{X}|\psi_{\xi}^{ml}\rangle &= \xi.\end{aligned}\tag{11}$$

The state $|\psi_x^{ml}\rangle$ is the generalization of the eigenstate $|x\rangle$ in the presence of a minimal length, in the sense that $|\psi_x^{ml}\rangle$ describes a system localized around x with the best possible resolution (which is not infinite in the present case). Indeed, $|\psi_x^{ml}\rangle \rightarrow |x\rangle$ when $\Delta x_{min} \rightarrow 0$, as we shall see. It is this property that makes them useful in trying to answer the above raised problems.

In what follows, we shall try to find an expression for these states in momentum representation.

3.1.1 Naïve Approach

The first attempt, which we call the “naïve approach” for reasons that will become apparent in a moment, would be to repeat the Kempf, Mangano, and Mann (KMM) prescription

⁶An hermitean operator is an observable iff its eigenvectors form a basis of the Hilbert space.

described in [13]. In their case, the starting point is the commutation relation given by Eq. 2, which leads to the uncertainty relation

$$\Delta x \geq \frac{\hbar}{2\Delta p}(1 + \beta\langle P \rangle^2) + \frac{\hbar\beta}{2}\Delta p.$$

It is then clear that Δx will be a minimum when the above relation is an equality, which occurs only if

$$\left[\hat{X} - \langle \hat{X} \rangle + \frac{\langle [\hat{X}, \hat{P}] \rangle}{2\langle (\Delta \hat{P})^2 \rangle} (\hat{P} - \langle \hat{P} \rangle) \right] |\psi_{squeezed}\rangle = 0. \quad (12)$$

We call states that obey this condition *squeezed states*, $\mathcal{H}_{squeezed}$ being their corresponding subspace. The squeezed state with the minimal value for Δx is the maximally localized state we are looking for.

We could then follow the same reasoning and, therefore, apply the same equation above to try to find our maximally localized states. Putting (5) into (12), projecting on $\langle p|$ and solving the differential equation we find

$$\psi_{squeezed}(p) \propto e^{-ik\langle \hat{X} \rangle} \exp \left[\frac{\Delta x}{\hbar \Delta p} \left(\frac{e^{-\beta p^2} - 1}{2\beta} + \hbar k \langle \hat{P} \rangle \right) \right], \quad (13)$$

where

$$\hbar k(p) = \int_0^p e^{-\beta q^2} dq.$$

Remembering that the Gaussian state is exactly the state for which the uncertainty relation is an equality, we see that $\psi_{squeezed}(p)$ is the generalized Gaussian wave function in the presence of a minimal length. Indeed, it is easy to see that we recover the familiar Gaussian function in the limit $\beta \rightarrow 0$.

Moreover, the oscillatory factor $e^{-ik\langle \hat{X} \rangle}$ in Eq. (13) has the form of a plane wave, which allows us to interpret $k(p)$ as a wave-vector. We thus recover the relation (3) postulated in [14], as it was our initial goal.

We can use this expression for $k(p)$ to estimate a relation between the parameter β and the minimal length Δx_{min} . To begin with, we note that, when $p \rightarrow \infty$, the wavelength λ of the wave associated to the particle approaches a minimum λ_{min} , which must be greater than or equal to Δx_{min} . Hence,

$$\frac{2\hbar\pi}{\Delta x_{min}} \geq \frac{2\hbar\pi}{\lambda_{min}} = \lim_{p \rightarrow \infty} \hbar k(p) \rightarrow \int_0^\infty e^{-\beta q^2} dq = \frac{\sqrt{\pi}}{2\sqrt{\beta}},$$

which implies

$$\Delta x_{min} \leq 4\hbar\sqrt{\pi\beta}.$$

From this and (8) it follows that

$$\sqrt{\frac{e}{2}} \leq \frac{\Delta x_{min}}{\hbar\sqrt{\beta}} \leq 4\sqrt{\pi}, \quad (14)$$

i.e., $\Delta x_{min} = a\hbar\sqrt{\beta}$, with

$$1, 17 \approx \sqrt{\frac{e}{2}} \leq a \leq 4\sqrt{\pi} \approx 7, 09. \quad (15)$$

From (13), a maximally localized state would be obtained by imposing that $\Delta x = \Delta x_{min}$ (which implies $\langle \hat{P} \rangle = 0$, see Eq. (8)). Thus,

$$\psi_{\xi}^{ml}(p) \propto e^{-ik\xi} \exp\left(\frac{\Delta x_{min}}{\hbar \Delta p^{min}} \frac{e^{-\beta p^2} - 1}{2\beta}\right) \quad (naïve \text{ result}) \quad (16)$$

where Δp^{min} is the value of Δp that minimizes Δx ⁷. Equation (16) would, then, be the maximally localized state we are looking for.

However, as it was pointed out in [18], the KMM prescription is not appropriate to construct maximal localization states for commutation relations like Eq. (5). So, the result (16) is not correct. The flaw lies in the premise that we must look for maximally localized states among squeezed states, as was done in [13]. While this prescription works well with their assumed commutation relation, in our case the right-hand side of the uncertainty relation is a complicated expression whose explicit form we do not know, and whose value depends on the state on which it is evaluated. It may well be possible to find a state outside $\mathcal{H}_{squeezed}$ (i.e., a state for which Eq. (6) is satisfied as an *inequality*), but whose Δx is still smaller than $\inf\left\{\langle \psi | \sqrt{(\Delta \hat{X})^2} | \psi \rangle, |\psi\rangle \in \mathcal{H}_{squeezed}\right\}$.

3.1.2 The DGS Approach

Having detected the problem of the KMM prescription when applied to general commutation relations, Detournay, Gabriel and Spindel (DGS) [18] go on to define an alternative and more abragent one, which is suitable to our particular case and which we now briefly describe.

Our goal is to find the states $|\psi_{\xi}^{ml}\rangle$ belonging to a certain subspace \mathcal{H}_{phys} ⁸ of the physical states and satisfying

$$(\Delta x_{min})^2 = \min\{\langle \psi_{\xi}^{ml} | \hat{X}^2 - \xi^2 | \psi_{\xi}^{ml} \rangle\} \equiv \mu^2, \quad (17)$$

where

$$\xi = \langle \psi_{\xi}^{ml} | \hat{X} | \psi_{\xi}^{ml} \rangle. \quad (18)$$

In momentum representation, this corresponds to

$$\left[-\left(\hbar e^{\beta p^2} \partial_p\right)^2 - \xi^2 + 2a\left(i\hbar e^{\beta p^2} \partial_p - \xi\right) - \mu^2\right] \psi_{\xi}^{ml}(p) = 0, \quad (19)$$

where a is a Lagrange multiplier introduced to account for Eq. (18). Imposing certain conditions to guarantee that the state indeed belongs to \mathcal{H}_{phys} ⁹ we arrive at the solution

$$\psi_{\xi}^{ml}(p) \propto e^{-i\xi k(p)} \sin\left[n\Delta x_{min}\left(k(p) + \frac{1}{2\hbar}\sqrt{\frac{\pi}{\beta}}\right)\right], \quad (20)$$

with $n \in \mathbb{N}$ and

$$\Delta x_{min} = \hbar\sqrt{\beta\pi}. \quad (21)$$

⁷ Δp^{min} has nothing to do with a minimal uncertainty in the momentum. The notation is admittedly bad, but we cannot think of a better one.

⁸Note that \mathcal{H}_{phys} is larger than $\mathcal{H}_{squeezed}$.

⁹We add, for example, the condition that $\langle \psi_{\xi}^{ml} | \hat{V}(p) | \psi_{\xi}^{ml} \rangle$ is finite for some unbounded observable \hat{V} , e.g. the energy. In such case, all we are saying is that the energy of a physical state must be finite.

This gives us an *exact* relation between the parameter β and the minimal length. Notice that $\frac{\Delta x_{min}}{h\sqrt{\beta}} \approx 1.77$, in agreement with Eq. (15). Notice, moreover, that we have again an exponential with the form of a plane wave, with $k(p)$ playing the role of the wave-number, as it should be.

Finally, it is easy to see that $\lim_{\beta \rightarrow 0} \psi_{\xi}^{ml}(p) \propto e^{-i\xi \frac{p}{h}}$, which proves that $|\psi_{\xi}^{ml}\rangle$ is indeed a generalization of the eigenvectors $|x\rangle$, as was claimed in the beginning of this section.

4 Discussions

We have seen that the existence of a minimal length raises many conceptual problems related to how to recover information on the localization of the system in space. They emerge because, in ordinary QM, all these informations are obtained by use of the eigenvectors $|x\rangle$ of \hat{X} , and these vectors do not belong to the Hilbert Space of the theory when a minimal length is present.

However, we have shown above that the maximally localized states are a generalization of these $|x\rangle$ to the formalism we now develop, and this might enable us to try to use them to answer the conceptual questions posed above. This is what we will do now.

4.1 Measurements in Quantum Mechanics

The first, and most fundamental of these questions, refers to the meaning of the expression “position measurement” in the presence of a minimal length. Let us investigate further what is really meant by this expression to understand where the problem is, and to try to solve it.

Let \hat{A} be the observable associated to some physical quantity \mathcal{A} , σ_A be its spectrum, \mathcal{H}_a be the eigenspace related to the eigenvalue $a \in \sigma_A$, and \hat{P}_a be the orthogonal projector onto \mathcal{H}_a .

In Quantum Mechanics, we are familiar with the notion that a measurement of \mathcal{A} is a perturbation acting on the system that causes it to collapse to one of the eigenspaces \mathcal{H}_a , whose corresponding eigenvalue is, then, the value obtained in the measurement. To the measurement process it is associated the projector $\hat{P}_a = \sum_{i=1}^{k_a} |a^i\rangle\langle a^i|$, k_a being the degeneracy of the obtained eigenvalue.

But there is a subtlety involved in this definition: it is valid only in the restricted case of a *sufficiently selective measurement*, i.e. a measurement whose result is a *single* value a . In general, however, a measurement causes the system to collapse not to a single eigenspace, but to a certain union of them, and the projector associated to the process is generally written as $\hat{P} = \sum_{a \in \sigma_A} c_a \hat{P}_a$, where c_a is related to the “intensity” each eigenvalue is obtained in the measurement.

All this becomes clearer (and more evident) when \hat{A} has a continuous spectrum. In this case, it is impossible to obtain a single value in a measurement, for it would require a device with an *infinite resolution*, and such thing does not exist¹⁰. In such case, every measurement results in an interval $I_a^{\delta a}$ centered in a and of length δa , and we then loosely say that a is the value measured with a resolution $(\delta a)^{-1}$. The associated projector is given by

$$\int_{-\infty}^{\infty} c(a') \hat{P}_{a'} da', \quad (22)$$

¹⁰This is indicated by the very formalism of the theory. If there were such a device, the state of the system afterwards would be a vector $|a\rangle$ that does not belong to \mathcal{H} , as we already pointed out for the particular case of position eigenvectors $|x\rangle$.

where $c(a')$ is a function whose exact expression depends on the measuring apparatus, but which, in general, has a maximum at a and is approximately zero when $|a' - a| \gtrsim \delta a/2$. In the case of a position measurement, say, performed with a photographic plate that becomes brighter in the region with which the incident particle interacts, the $c(x')$ gives the intensity of this brightness as a function of the coordinate x' of the plate. Since we can hardly tell the exact form of this function, we formulate the working hypothesis that the apparatus is a *perfect filtering device*, which means the collapsed wave-function is exactly zero outside $I_a^{\delta a}$, and remains unaltered otherwise. That is,

$$c(a') = \begin{cases} 0, & a' \notin I_a^{\delta a} \\ 1, & a' \in I_a^{\delta a} \end{cases} \quad (23)$$

Under these conditions, (22) assumes the form

$$\int_{a-\frac{\delta a}{2}}^{a+\frac{\delta a}{2}} \hat{P}_{a'} da' \quad (24)$$

which is exactly the form of a projector associated to a measure of a degenerate eigenvalue, whose eigenvectors are $\{|a'\rangle, a' \in I_a^{\delta a}\}$. This allows us to reinterpret an *insufficiently selective measurement* in the following way.

Let the interval $I_a^{\delta a}$ be given in advance. Define an operator $\mathcal{O}_a^{\delta a}$ whose eigenvalues are 0 and 1, and whose eigenvectors associated with the eigenvalue 1 are the vectors for which a measurement of \hat{A} yields values contained in $I_a^{\delta a}$, while the eigenvectors associated to 0 are the vectors for which a measurement of \hat{A} yields values that are not in such interval. Since \hat{A} and $\mathcal{O}_a^{\delta a}$ share the same eigenvectors — only the associated eigenvalues change —, \hat{A} being an observable implies that $\mathcal{O}_a^{\delta a}$ is an observable, and, therefore, there is a physical quantity associated to it. Now, note that the projector onto the eigenspace \mathcal{H}_1 of this operator is precisely (24), which means that to measure \hat{A} and obtain $I_a^{\delta a}$ is equivalent to measure $\mathcal{O}_a^{\delta a}$ and obtain 1.

This means that we can interpret a position measurement as a question of “whether or not the system is localized in a certain predefined region of space”. A position measurement apparatus with resolution $(\delta x)^{-1}$ is, thus, nothing more than a set of detectors with length δx , each of which changes its state (say, emits a beep) when it interacts with the particle.

4.2 Position Measurement with a Minimal Length

In the presence of a minimal length, the eigenvalues of the position operator do not belong to the Hilbert Space anymore, and, as a consequence, \hat{X} loses its status of an observable. Since there are no more eigenspaces \mathcal{H}_x , nor projectors P_x onto such spaces, it is simply impossible to speak of a position measurement — even of a measurement with $\Delta x > \Delta x_{min}$, which would not violate our initial postulate.

Nevertheless, we should still find some way to work around this problem and recover information on the spatial configuration of the system, otherwise the theory would be incomplete, even incapable of describing many systems in which the essential informations are on spatial localization, like a particle moving in a cloud chamber, for example.

This can indeed be done using the alternative interpretation of a position measurement given above. We say that to “measure” the position of a system with the greatest possible resolution is to make it interact with a properly regulated apparatus (an array of detectors each with length Δx_{min}) that indicate if this interaction occurred in a previously determined

spatial region, centered in ξ and with length Δx_{min} , or not¹¹. In case affirmative, the perturbation causes the system to collapse to the respective Maximal Localization State, $|\psi_{\xi}^{ml}\rangle$, and the “projector” associated to the “measurement” is

$$\hat{P}_{\xi}^{ml} = |\psi_{\xi}^{ml}\rangle\langle\psi_{\xi}^{ml}|. \quad (25)$$

Strictly speaking, we cannot say this is a measurement¹², since we have not presented an observable associated with it. But it definitely is a way to recover information on the spatial localization of a system, and this was, after all, all we were looking for.

Under one aspect, this definition is even less problematic than the one of ordinary QM. In that case, the correct form of the projector associated to the measure is (22), and the coefficients $c(x')$ are unknown. We generally solve this difficulty by making the assumption that leads to (24), but this assumption is physically incorrect, not only because there are no perfect filters in reality, but, much graver than that, because the action of this projector on a state may result in a discontinuous wave-function, which is absurd. In the present case no such problems arise: the “projector” is known to be (25), and the resulting state after the “measurement” is simply $|\psi_{\xi}^{ml}\rangle$.

4.3 Probability and Mean Values

The Maximal Localization States can also be used to calculate the probability for a certain system, whose state is described by $|\phi\rangle$, to be found around the point ξ when its position is “measured” with the greatest possible resolution. This probability is, of course, given by

$$\mathcal{P}_{\xi}^{ml} = |\langle\psi_{\xi}^{ml}|\phi\rangle|^2.$$

The function

$$\phi(\xi) \equiv \langle\psi_{\xi}^{ml}|\phi\rangle$$

is the generalization of the wave-function of ordinary Quantum Mechanics, and gives us an useful expression for the spatial distribution of the system.

There is yet another way to recover information on how the system is localized in space, which is by calculating the mean values related to the operator \hat{X} , like

$$\begin{aligned} \langle\hat{X}\rangle &= \langle\phi|\hat{X}|\phi\rangle \\ \langle(\Delta\hat{X})^2\rangle &= \langle\phi|(\hat{X} - \langle\hat{X}\rangle)^2|\phi\rangle. \end{aligned} \quad (26)$$

One might then ask: if \hat{X} is not an observable, does (26) still make sense? Of course we can still calculate these quantities (they are simply the matrix elements of an hermitean operator), but how could we interpret them? To answer this, we first note that, though \hat{X} loses its status of an observable, it still has the physical interpretation of being an operator associated to the position of the system¹³. That \hat{X} is not an observable simply tells us that we cannot measure this position anymore (even to formulate the statement “one cannot measure position because \hat{X} is not an observable” we already assume the relation between

¹¹For a similar interpretation, generalized to the covariant case (in which we measure the localization of the system not only in Space, but in Spacetime) see [19].

¹²Hence the quotes in the above paragraphs.

¹³An operator does not need to be an observable, not even hermitean(!), to have a physical interpretation. Take the creator and annihilator operators, \hat{a} and \hat{a}^{\dagger} , as examples.

this operator and the spatial localization of the system.). Therefore, these mean values do tell us something about the way the system is distributed in space.

Of course, we cannot interpret $\langle \hat{X} \rangle$ as the mean of the results obtained in a series of measurements, nor can we say that Δx is the mean deviation of these measurements. But we can use a less statistical and more geometrical interpretation, associating $\langle \hat{X} \rangle$ to the point around which the system is localized, and $\langle (\Delta \hat{X})^2 \rangle$ to how much it is spread around this point.

The existence of a Δx_{min} guarantees, therefore, that the physical system is never concentrated around a point with an arbitrary precision, in agreement with our interpretation of Δx_{min} as a maximal resolution to the localization of the system.

5 Minimal Length as a Regulator in QFT

We now turn our attention to an extremely interesting property of a minimal length: its role as a natural regulator of Quantum Field Theory.

As already noted before, the notion of a particle perfectly localized in a point is not well defined in Quantum Mechanics, for the vector associated to such state does not belong to the space of physical states. In the non-relativistic theory this can be worked around by doing the calculations with the states $|\xi^{\Delta x}\rangle$, and taking the limit $\Delta x \rightarrow 0$ afterwards. We say that this is only a workaround because, though we recognize that the notion of precise localizability leads us to problems, we end up taking the limit that corresponds to such case, thus running from the responsibility of really solving the problem in its roots.

Of course, this can only be a temporary solution, and, indeed, this problematic notion of perfect localizability ends up giving rise to the much graver problem of the divergences that plague Quantum Field Theory. Indeed, it is now widely accepted that these divergences appear because, in QFT, the interactions are considered to occur in a single *spacetime point*, or, in other words, because we evaluate the propagators as $G(x, x') = \langle x | \hat{G} | x' \rangle$ [20].

Again, this difficulty can be bypassed by regularization methods similar to the procedure we did in the non-relativistic case, i.e. by introducing a restriction to the localizability of the interactions, doing the calculations and then taking the appropriate limit to return to the case of interactions occurring at a point. But, again, this mere workaround shows itself as unsatisfactory, because it does not work for the gravitational interaction — to mention only one strong reason.

In this work we attack the very cause of these problems by introducing the notion of a non-zero minimal observable length, thus eliminating the notion of a perfectly localized event. In such case, the states $|\psi_{\xi}^{ml}\rangle$ maximally localized around a point ξ are not eigenstates of the position operator, but are given by (16), and the propagator must then be defined as $G(\xi, \xi') = \langle \psi_{\xi}^{ml} | \hat{G} | \psi_{\xi'}^{ml} \rangle$. This introduces a non-local aspect to the field interactions, and we therefore expect the theory to be regular, i.e. divergence-free.

In [17] it was shown that, for a scalar field theory with a ϕ^4 self-interaction, this is, indeed, what happens: due to the presence of a minimal length, the Feynman diagrams do not diverge at any order of expansion!

In the following, we will also show that the Casimir Energy calculated from the formalism we have developed is finite. This is interesting because the infinities that appear in the calculations of the Casimir Effect are one of the most classic examples of the divergences that plague QFT, and it is a very strong result that the presence of a minimal length regularizes it. Moreover, due to high precision experimental measurements achieved, the Casimir effect

may provide experimental constraints on the value of the minimal length [21, 22].

5.1 Casimir Effect

Let us consider the problem of the one-dimensional Casimir Effect generated by the Electromagnetic field in the presence of two parallel conducting “plates” separated by a distance L .

In QFT, the E-M field is described by an infinite number of uncoupled harmonic oscillators, each with ground state energy given by

$$E_p = \frac{\hbar\omega_p}{2} = \frac{pc}{2},$$

where p is the linear momentum of the photon associated to such oscillator. Inverting Eq. (3) to write this in terms of the wave number k , we have

$$E_k = \frac{c}{2\sqrt{\beta}} \operatorname{erf}^{-1} \left(\frac{2\hbar\sqrt{\beta}}{\pi} k \right).$$

The total energy of the vacuum state of the field is, therefore, the sum of these energies over all possible k , that is, over all allowed points in k -space.

Now, since the wave-number k is bounded from above by

$$k_{max} = \frac{\sqrt{\pi}}{2\hbar\sqrt{\beta}}$$

we can see that the result of this vacuum energy must be finite. This becomes even clearer when we impose the boundary conditions, resulting in the requirement that

$$k_n = \frac{n\pi}{L}.$$

Here, n is a positive integer, but since there is a k_{max} , there must also exist an n_{max} , which must be the greatest integer smaller than

$$\frac{1}{A} \equiv \frac{L}{2\hbar\sqrt{\pi\beta}}.$$

The Casimir energy, consisting in the difference between the vacuum energy with and without the boundary conditions, can then be written as¹⁴

$$E_{cas} = \frac{c}{2\sqrt{\beta}} \left[\sum_{n=0}^{n_{max}} \operatorname{erf}^{-1}(An) - \int_0^{\frac{1}{A}} \operatorname{erf}^{-1}(Ax) dx \right], \quad (27)$$

and the finiteness of the result becomes clear, since the integral converges (!) and the sum is over a finite number of terms. This shows that a minimal length indeed acts as a regulator of QFT, as discussed above.

We can calculate the energy explicitly using the Euler-Maclaurin formula, according to which

$$\sum_{k=0}^n f(k) = \int_0^n f(x) dx + \frac{1}{2}(f(n) + f(0)) + \sum_{k=1}^N \frac{B_{2k}}{(2k)!} (f^{(2k-1)}(n) - f^{(2k-1)}(0)) + R_N \quad (28)$$

¹⁴We have added $n = 0$ to the sum for convenience, which does not alter the result since $\operatorname{erf}^{-1}(0) = 0$.

where B_{2k} are the Bernoulli numbers, N is an arbitrary integer and R_N is the error of the approximation for a given N . However, it is difficult to analyse the result obtained using this formula, since we will end up with a functional dependence on n_{max} , and we do not have a closed expression for this variable as a function of L , β and other parameters of the problem.

To work around this difficulty and perform the calculations, we will use a small trick. Note that, since the integral in (27) converges, for every $\epsilon > 0$ we choose there is a $\delta \in (0, 1/A)$ such that $\int_0^\delta \text{erf}^{-1}(Ax)dx$ differs from the desired integral by less than ϵ . Thus, given an $\epsilon > 0$, let us define $f(x)$ as a function that equals $\text{erf}^{-1}(Ax)$ for all $x \leq \max\{\delta, n_{max}\}$, goes smoothly to zero for $x > \max\{\delta, n_{max}\}$, and equals zero for every $x \geq 1/A$. We then have ¹⁵

$$\sum_{n=0}^{n_{max}} \text{erf}^{-1}(An) = \sum_{n=0}^{n_{max}} f(n) = \sum_{n=0}^{\infty} f(n)$$

and

$$\int_0^{1/A} \text{erf}^{-1}(Ax)dx - \int_0^{\infty} f(x)dx < \epsilon,$$

and we can therefore write

$$E_{cas} = \frac{c}{2\sqrt{\beta}} \left[\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(x)dx \right]$$

up to an error ϵ that we control. This extension of the domain of the sum/integral to infinite was our motivation to introduce the new function $f(x)$. Since $f(x) = 0 \forall x \geq 1/A$, and because $f(x) = \text{erf}^{-1}(Ax)$ in a neighborhood of $x = 0$, Eq. (28) leads us to

$$E_{cas} = -\frac{c}{2\sqrt{\beta}} \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{d^{2k-1}}{dx^{2k-1}} \text{erf}^{-1}(Ax) \Big|_{x=0},$$

and we thus arrive at a powerful expression that allows us to compute the Casimir energy to any order we want.

Evaluating the derivatives and using $B_2 = 1/6$ and $B_4 = -1/30$ we obtain

$$E_{cas} = -\frac{\hbar\pi c}{24L} + \frac{\hbar^3\pi^3 c}{720L^3}\beta + \mathcal{O}(\beta^2). \quad (29)$$

The Casimir force between the plates, given by $F_{cas} = -\partial E_{cas}/\partial L$, is, therefore,

$$F_{cas} = -\frac{\hbar\pi c}{24L^2} + \frac{\hbar^3\pi^3 c}{240L^4}\beta + \mathcal{O}(\beta^2). \quad (30)$$

Note that, to order β^0 , we recover the well-known Lüscher result for the (1+1)-dimensional Casimir force [23, 24].

Note that the minimal length introduces a repulsive factor to the Casimir force. When the distance between the plates are much greater than the minimal length (which is proportional to $\hbar\sqrt{\beta}$, as we have seen) this repulsive factor is very small, but it becomes more significant when $L \approx \hbar\sqrt{\beta}$.

¹⁵We can extend the sum to infinity because, by definition of n_{max} , $n > n_{max} \Rightarrow n > 1/A \Rightarrow f(n) = 0$.

6 Conclusions

We have proposed a modification of the algebra of the quantum operators in order to introduce a $\Delta x_{min} > 0$ in the formalism, which acts as a minimal observable length. Our aim was to understand the consequences of the existence of such effect, and what Quantum Mechanics is like when it is present. This is interesting because a minimal length acts as a natural regulator of Quantum Field Theories, so that if we manage to extend the formalism here developed to QFT we end up with a divergence-free theory.

Our choice of the algebra given in Eq. (5) was motivated by the work of [14], in which the authors start by postulating a dispersion relation like (3) in order to study the Casimir Effect in the presence of a minimal length. However, we adopt here the most rigorous approach of [13], starting with a modification of the algebra of the operators (which is the very starting point of Quantum Mechanics) and obtaining Eq. (3) as a consequence of the formalism. The advantage is that, in taking this most fundamental starting point, we recognize the many subtleties related to the existence of a minimal length, most of which stemming from the fact that there is no position representation in the formalism. Moreover, we not only showed that the minimal length regularizes the Casimir Energy, we have also calculated it.

There is also another consequence of the existence of a minimal length that we have not discussed in this work, but which is also worth mentioning. Let ℓ_{min} be a minimal length, and suppose we have a certain object of length ℓ_{min} , as measured in the rest frame S of the object. Then, if we accept the results of Special Relativity (in particular, Lorentz contraction), a frame S' moving with respect to S would measure the length of the object to be $\ell < \ell_{min}$, which is absurd. This means that, if we accept the existence of a minimal length, the postulates of the theory of Relativity must be modified to avoid this contradiction.

In [25, 26, 27, 28, 29] it was shown that this can be done by including a postulate according to which the minimal length ℓ_{min} is also a universal constant of nature, together with c . We then have a theory of Relativity with two universal constants — hence the name Doubly Relativity. The resulting expression for the “generalized Lorentz contraction” is such that the length ℓ of a moving object is always greater than or equal to ℓ_{min} , regardless of its speed relative to the observer, and the above mentioned inconsistency is, therefore, solved.

In this work, we have restricted ourselves to the simple one-dimensional case. This can be generalized to the tridimensional case if we impose a commutation relation of the form

$$[\hat{X}_i, \hat{P}_j] = i\hbar e^{\beta \hat{\mathbf{P}}^2} \delta_{ij}.$$

The operators in the $|\mathbf{p}\rangle$ representation can, then, be written as

$$\langle \mathbf{p} | \hat{P}_i | \psi \rangle = p_i \psi(\mathbf{p}),$$

$$\langle \mathbf{p} | \hat{X}_i | \psi \rangle = i\hbar e^{\beta \mathbf{p}^2} \frac{\partial}{\partial p_i} \psi(\mathbf{p}),$$

thence we can see that \hat{X}_i and \hat{X}_j do not commute, showing that our proposal is intimately related to the proposals of quantizing gravity by postulating a non-commutative geometry, as in [30, 31].

Finally, it would be interesting to extend this formalism from the non-relativistic case to QFT. This is not simple, because we must take into account not only the change in the commutation relations, but also in the invariance group of the theory, which is not the

Poincaré group anymore (because of the comments made a couple of paragraphs above), but a more complicated extension of it. It would also be interested to perform the calculations of the Casimir Force for the three-dimensional case, and compare the results obtained with the experimental data, which would give us an upper bound for the minimal length.

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